Basics Related to Relative Brauer Groups
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1 Splitting Fields

Definition. *Consider a field extension $K|k$ and a central simple algebra A over k. We say that K is a splitting field for A if $A \otimes_k K \cong M_n(K)$ (as K-algebras).*
We will often write $A_K = A \otimes_k K$.

Notice that $\overline{k}$ splits every central simple algebra over k. Also, notice that if K splits A, then K will also split any central simple algebra similar to A. Therefore, it makes sense to define the relative Brauer Group:

Definition. *Fix a field extension $K|k$. We define the relative Brauer group* $\mathrm{Br}(K|k)$ *to be the subgroup of* $\mathrm{Br}(k)$ *of central simple algebras split by K.*

By the remark above, $\mathrm{Br}(K|k)$ is a well-defined subset of $\mathrm{Br}(k)$, but we need to show that $\mathrm{Br}(K|k)$ is actually a subgroup of $\mathrm{Br}(k)$. In fact, we will also show that K can be assumed to be a finite Galois extension of k.

Proposition. $\mathrm{Br}(K|k)$ *is actually a subgroup of* $\mathrm{Br}(k)$..

Proof. We know that A_K is a central simple K-algebra, so we have a map $\mathrm{Br}(k) \to \mathrm{Br}(K)$ via $[A] \mapsto [A_K]$ (where $[\cdot]$ means the similarity class). To see that the above map is actually a homomorphism, notice that

$$(A \otimes_k B) \mapsto (A \otimes_k B) \otimes_k K \cong (A \otimes_k K) \otimes_K (B \otimes_k K) = A_K \otimes_K B_K$$

so our map is actually a homomorphism.

The kernel of the above homomorphism is exactly all central simple A-algebras that are split by K. Therefore, by definition, $\mathrm{Br}(K|k)$ is a subgroup of $\mathrm{Br}(k)$, as desired. □

In fact, we can give another characterization of the relative Brauer group which is more useful when thinking about maximal subfields. We will give this characterization after introducing some basic definitions.

Definition. *Recall that the dimension of a central simple algebra A over k is a square. We define the degree of A over k to be $\sqrt{\dim_k(A)}$.*

We can now give another characterization of $\mathrm{Br}(K|k)$:

Theorem. *Let A be a central simple algebra over k. If A contains a subfield K with $[K : k] =$* $\deg(A)$*, then K splits A.*

Proof. Recall from last time that $A \otimes_k A^{op} \cong \mathrm{End}_k(A)$. If $K \subseteq A$ (with K as in the statement), then $K \subseteq A^{op}$ because K is commutative. In other words, we have an injection

$$\iota : A \otimes_k K \to A \otimes_k A^{op} \cong \mathrm{End}_k(A).$$

Now, since $K \subseteq A$, we can view A as a K-vector space with K acting on A via right multiplication. Now, recall that $A \otimes A^{op} \cong \mathrm{End}_k(A)$ via

$$\sum_i a_i \otimes_k b_i \mapsto \left(x \mapsto \sum_i a_i x b_i\right)$$

Therefore, since A is a right K-vector space,

$$\iota\left(\sum_i a_i \otimes_k c_i\right) = (x \mapsto a_i x_i c_i)$$

is actually a K-endomorphism of A. In short, we have shown that

$$A \otimes_k K \subseteq \mathrm{End}_K(A).$$

Since $\mathrm{End}_K(A) \cong M_n(K)$, it suffices to show that $A \otimes_k K$ surjects onto $M_n(K)$. To be more careful,

$$\mathrm{End}_K(A) \cong M_n(K) \cong M_n(k) \otimes_k K \cong \mathrm{End}_k(A) \otimes_k K,$$

so $n = \deg_k(A)$. However, since $[K : k] = \deg_k(A)$, $\dim_K(M_n(K)) = \deg_k(A)^2$ and $\dim_K(D \otimes_k K) = \deg_k(A)^2$. Hence, we have shown there is a K-algebra isomorphism $A \otimes_k K \cong \mathrm{End}_K(A) \cong M_n(K)$, as desired. □

There is also a converse to the above result.

Theorem. *Let A be a central simple algebra over k. Assume that $[K : k] = \deg(A)$ and K splits A. Then there is a unique (up to isomorphism) central simple k-algebra A' similar to A so that* $\deg(A) = \deg(A')$ *and $K \subseteq A'$.*

Proof. By Wedderburn we can write $A = M_n(D)$. Since $[D] = A$, we know that K also splits D. Therefore, $D \otimes_k K \cong M_m(K)$ where $m = \deg_k(D)$. It follows that

$$D^{op} \otimes_k K \cong (D \otimes_k K)^{op} \cong M_m(K)^{op} \cong M_m(K),$$

so K also splits D^{op}. Furthermore, consider $V = K^m$. The above string of isomorphisms tells us that if we embed V into the diagonal of $M_m(K)$, V is a left module of D^{op} (equivalently V is a right module of D). Thus, by a slight generalization of Josh's result,

$$\mathrm{End}_{D^{op}}(V) = M_t(D^{op}) \cong M_t(D)^{op} \cong M_t(D)$$

for $t = \dim_{D^{op}} V$. Notice that any element of K induces a D^{op}-endomorphism of V given by multiplication. Thus, $K \subseteq M_t(D)$.

On the other hand,

$$\dim_k V = m[K : k] = t \dim_k(D^{op}) = t \dim_k(D).$$

Hence,

$$t^2 \dim_k D = \frac{(mn)^2}{\dim_k D} = n^2.$$

Thus, $M_t(D)$ has dimension n^2 and contains a copy of K. Furthermore, since $A \cong M_n(D)$, we know that A is similar to $M_t(D)$, so $M_t(D)$ is the desired central simple algebra. □

So far, we do not know that there is any meaningful relationship between $\mathrm{Br}(k)$ and $\mathrm{Br}(K|k)$. Our goal will be to show that $\cup_{K|k \text{ finite}} \mathrm{Br}(K|k) = \mathrm{Br}(k)$. Before achieving this goal, we need to prove more results.

2 Skolem-Noether

Theorem. ***Skolem-Noether.*** *Let A, B be finite dimensional simple k-algebras with B central. If $f, g : A \to B$ are two k-algebra homomorphisms then there exists a unit $b \in B$ so that*

$$g(a) = bf(a)b^{-1}$$

for all $a \in A$.

Theorem. *Notice that since B is central, B^{op} is central as well. Therefore, we know that $A \otimes_k B^{op}$ is also simple. Thus, for f and g, we can associate a $A \otimes_k B^{op}$-module structure on B via*

$$(a \otimes b)_f \cdot x = f(a)xb$$

and similarly for g. We denote B_f and B_g the structures associate with f and g respectively.

Notice that $\dim_k B_f = \dim_k B_g$, so $B_f \cong B_g$ as $A \otimes_k B^{op}$-modules. ***SHOULD I DESCRIBE WHY THIS IS TRUE*** *For ease of notation, let $\phi : B_f \to B_g$ be the isomorphism and set $b = \phi(1)$. It follows that for all $x \in B$*

$$\phi(x) = \phi((1 \otimes x)_f \cdot 1) = (1 \otimes x)_g \cdot \phi(1) = bx.$$

By the same argument applied to ϕ^1, we get that $\phi^{-1}(x) = b'x$ where $b' = \phi^{-1}(1)$. Thus,

$$x = \phi \circ \phi^{-1}(x) = bb'x$$

for all $x \in B$. If we set $x = b$, then we get $bb' = 1$. By the same argument we get $b'b = 1$, so b is actually a unit in B. It follows that

$$bf(a) = \phi(f(a)) = \phi((a \otimes 1)_f \cdot 1) = (a \otimes 1)_g \phi(1) = g(a)b,$$

so the result follows.

Corollary. *Let A be a central simple algebra over k. Then every k-algebra automorphism of A is inner.*

Proof. If $g : A \to A$ is a k-algebra automorphism, then the Skolem-Noether Theorem applied to $f = \mathrm{id}$ gives us the desired result. □

3 Double Centralizer

Definition. *Let A be a central simple algebra over k and $B \subseteq A$ a simple subalgebra. We define the centralizer to be*

$$Z_A(B) = \{x \in A : xb = bx \ \forall b \in B\}.$$

Theorem. ***Double Centralizer Theorem.*** *Let A be a central simple k algebra with $\dim_k(A) = n$ and $B \subseteq A$ a simple subalgebra with $\dim_k(B) = m$. Then*

1. *$Z_A(B) \otimes_k M_m(k) \cong A \otimes B^{op}$.*
2. *$Z_A(B)$ is a simple A-subalgebra with dimension $\dim_k Z_A(B) = n/m$.*
3. *$B = Z_A(Z_A(B))$.*

Proof. We shall prove 3 first. By definition, we know that $B \subseteq Z_A(Z_A(B))$. Furthermore, by part 2 we know that $Z_A(B)$ is a simple A-subalgebra and $\dim_k Z_A(B) = n/m$. Therefore, by part 2, again, we find that $Z_A(Z_A(B))$ is a simple A-subalgebra with dimension $\dim_k Z_A(Z_A(B)) = n/(n/m) = m = \dim_k B$. Since $B \subseteq Z_A(Z_A(B))$, by the dimension considerations above, $B \cong Z_A(Z_A(B))$, as needed.

We shall now prove 2. We know that $A \otimes_k B^{op}$ is a central simple algebra. By part 1, it follows that $Z_A(B) \otimes_k M_m(k)$ is also a central simple algebra. However, since $M_m(k)$ is a central simple k-algebra, every two-sided ideal of $Z_A(B) \otimes_k M_m(k)$ is of the form $\mathfrak{a} \otimes_k M_m(k)$ for an ideal $\mathfrak{a} \subseteq Z_A(B)$. As we saw, $Z_A(B) \otimes_k M_m(k)$ is simple, so that means that $\mathfrak{a} \otimes_k M_m(k)$ is trivial. In other words, $\mathfrak{a}$ is trivial, so $Z_A(B)$ is simple. $Z_A(B)$ is a sub-algebra of A by definition.

Now, by using the isomorphism $Z_A(B) \otimes_k M_n(k)) \cong A \otimes B^{op}$, we get that

$$\dim_k(Z_A(B))m^2 = nm,$$

so $\dim_k(Z_A(B)) = n/m$, as desired.

It remains to prove part 1. We will consider two different embeddings:

$$f, g : B \to A \otimes_k \operatorname{End}_k(B)$$

via $f(b) = b \otimes \mathrm{id}_B$ and $g(b) = 1 \otimes \lambda_b$ where λ_b is left multiplication by b. Now, recall that $\operatorname{End}_k(B) \cong M_m(k)$, in particular, $\operatorname{End}_k(B)$ is a central simple algebra. By assumption, A is a central simple algebra, so

$$A \otimes_k \operatorname{End}_k(B)$$

is also a central simple algebra.

It follows by Skolem-Noether that f and g are conjugate. In other words, we can choose a unit $x \in A \otimes_k \operatorname{End}_k(B)$ such that

$$f(b) = xg(b)x^{-1}$$

for all $b \in B$. It follows that

$$Z_{A\otimes_k \operatorname{End}_k(B)}(f(B)) = xZ_{A\otimes_k \operatorname{End}_k(B)}(g(B))x^{-1}$$

as sets. By multiplying each side by x, we actually find that the centralizers are isomorphic:

$$Z_{A\otimes_k \operatorname{End}_k(B)}(f(B)) \cong Z_{A\otimes_k \operatorname{End}_k(B)}(g(B))$$

By definition of g and f, we can rewrite the above isomorphism as:

$$Z_A(B) \otimes_k \operatorname{End}_K(B) \cong Z_{A\otimes_k \operatorname{End}_k(B)}(f(B)) \cong Z_{A\otimes_k \operatorname{End}_k(B)}(g(B)) \cong Z_{A\otimes_k \operatorname{End}_k(B)}(A \otimes_k L(B))$$

where $L(B)$ is the image of multiplication on the left of B. However, from what Joshua also mentioned last time,

$$Z_{A\otimes_k \operatorname{End}_k(B)}(A \otimes_k L(B)) \cong A \otimes_k R(B^{op}) \cong A \otimes B^{op}.$$

By combining all of the isomorphism, we obtain the result we wanted. □

With this major result in place, we can spend some time thinking about subfields again; specifically, maximal subfields.

Corollary. *Let A be a central simple algebra over k of degree n. If $[K : k] = d$ and K is a subfield of A then d divides n and $Z_A(K)$ is a central simple algebra over K of degree $\deg_K Z_A(K) = n/d$.*

In particular, if $d = n$ then $Z_A(K) = K$ and, consequently, K is a maximal subfield of A.

Proof. Since $K \subseteq A$, we know that $Z_A(K)$ makes sense. Then, by part 2 of the double centralizer theorem, $Z_A(K)$ is a simple subalgebra of A of dimension

$$\dim_k Z_A(K) = n^2/d.$$

In particular, d divides n (also, we have the square because n is the degree, not the dimension). Since $[K : k] = d$, it follows that

$$\dim_K Z_A(K) = n^2/d^2,$$

as desired. □

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We can also give a partial converse to the above result:

Corollary. *Assume that D is a central division algebra over k. Also assume that $K \subseteq D$ is a maximal subfield then* $\dim_k K = n = \deg_k D$.

Proof. Assume that $K \subseteq D$ is a maximal subfield. Notice that $k \subseteq K$, for if $k \not\subseteq K$ then we could consider the compositum of k and K (which would be a bigger field containing K, a contradiction). Notice if $x \in Z_D(K) \setminus K$ then $K[a]$ is a field containing K that is strictly bigger, a contradiction (we are using the assumption that D is a division algebra in this step). Therefore, $K = Z_D(K)$, so by the previous corollary, $n = [K : k] = \dim_k K$, as needed. □

We will finally get to the result that allows us to relate the relative Brauer group and the absolute Brauer group:

Theorem. *Let D be a central division algebra over k. D contains a maximal subfield K which is a separable extension of k.*

Proof. If k is perfect, we are done. Therefore, assume that k is an infinite field of positive characteristic p. We will first show that there exists an element $a \in D \setminus k$ which is separable over k. Notice that if $\deg_k(D) = n$ is not a power of p, then some element in the maximal subfield K is separable. This is because, by the above corollary, $[K : k] = \dim_k K = n$ and because purely inseparable extensions must be of degree power of p. Now, assume that $n = p^\alpha$ and that $D \setminus k$ only contains purely inseparable elements. We also know that, for all $a \in D$, $a^n \in k$. With this in mind, pick a basis $e_1 = 1$, e_2, …, e_{n^2} of D over k. Then we can choose polynomials f_1, …, $f_{n^2} \in k[t_1, \dots, t_{n^2}]$ such that

$$(t_1 e_1 + \cdots + t_{n^2} e_{n^2})^n = f_1 e_1 + \cdots + f_{n^2} e_{n^2}.$$

However, as we mentioned above, $a^n \in k$ for all $a \in D$, so

$$(a_1 e_1 + \cdots + a_{n^2} e_{n^2})^n$$

is in the span of e_1 for all $(a_1, \dots, a_{n^2}) \in k^{n^2}$. It follows that

$$f_2(a_1, \dots, a_{n^2}) = \cdots + f_{n^2}(a_1, \dots, a_{n^2}) = 0$$

for all $a_i \in k$. Since k is infinite and f_i are polynomials, that means that each $f_i = 0$. In particular, each polynomial is 0 on $\overline{k}^{n^2}$. In other words, $a^n \in \overline{k}$ for all $a \in D \otimes_k \overline{k}$. We know that $D \otimes_k \overline{k} \cong M_n(\overline{k})$. However, notice that, with respect to the usual basis, $e_{11} \in M_n(\overline{k})$ has the property that $e_{11}^n = e_{11}$, so $e_{11} \in \overline{k}$ by the above argument. By looking at the determininant, e_{11} is a zero-divisor, so $e_{11} \notin \overline{k}$, a contradiction! Hence, there exists a separable element $a \in D \setminus k$.

With the existence of the separable element in place, we will finish the proof by induction on $\deg_k(D) = n$. If $n = 1$ then $D = k$, so the maximal subfield is k. It is certainly separable. If $n > 1$, consider $[k(a) : k] = d > 1$ for the a found above. By the above corollary, $Z_D(K(a))$ is a central division algebra over $k(a)$ and it has dimension

$$\dim_{k(a)} Z_D(k(a)) = (n/d)^2 < n.$$

By the inductive hypothesis, $Z_D(k(a)$ contains a maximal subfield K that is a separable extension of k. By the above corollary $[K : k(a)] = n/d$, so $[K : k] = n$. By the other above corollary, this means K is a maximal subfield of D. K is separable by assumption, as desired. $\square$

We now have all the tools in place to compare relative and absolute Brauer Groups:

4 Relating Relative and Absolute Brauer Groups

Theorem. $\mathrm{Br}(k) = \cup_K \mathrm{Br}(K|k)$ *where the union is taken over all finite Galois extensions of K.*

Proof. It is clear that the union is contained in $\mathrm{Br}(k)$, so we will just show the reverse inclusion. With this in mind, let A be a central simple k-algebra. Wedderburn tells us that $A \cong M_n(D)$ for a division algebra D. By the above theorem, we can find a maximal subfield K of D which is a separable extension of k with $[K : k] = d$. From the start of the talk, we know that

$$D \otimes_k K \cong M_d(K),$$

so

$$A \otimes_k K \cong (M_n(k) \otimes_k D) \otimes_k K \cong M_n(K) \otimes_K D_K \cong M_n(K) \otimes_K M_d(K) \cong M_m(K).$$

Since K is separable over k, the normal closure of K, K' is a finite Galois extension of k. Notice that

$$A \otimes_k K' \cong (A \otimes_k K) \otimes_K K' \cong M_m(K) \otimes_K K' \cong M_m(K').$$

Hence, $A \in \mathrm{Br}(K'|k)$ with K' a finite Galois extension of k, as desired. $\square$

Notice that, in particular, the above theorem says that every central simple algebra has a splitting field.

A very important part of the above theorem is that $\mathrm{Br}(k)$ is completely described by its finite Galois extensions. This suggests that the absolute Galois group of k will give a ton of information about k (and vice verse).